Approximation numbers via bracketing

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Abstract

Let B be the unit ball in $\mathbb{R}^n$, $m \in \mathbb{N}$ and $1 \leq p < \infty$. We define the weighted Sobolev space $E_{p,\sigma}^m(B)$ as the completion of $C_0^m(B) = \{f \in C^m(B) : \operatorname{supp} f \text{ compact}\}$ with respect to the norm

$$\|f \mid E_{p,\sigma}^m(B)\| := \left(\int_B |x|^{mp} \left(1 + |\log|x||\right)^{\sigma p} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{1/p}.$$

Then, if $\sigma > 0$, the embedding

$$\mathrm{id} : E_{p,\sigma}^m(B) \hookrightarrow L_p(B)$$

is compact. In case of Hilbert spaces, $p = 2$, Triebel obtained in [3] sharp results for the corresponding approximation numbers

$$a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}} & , \text{ if } \sigma > \frac{m}{n} \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}} & , \text{ if } \sigma = \frac{m}{n} \\ k^{-\sigma} & , \text{ if } 0 < \sigma < \frac{m}{n}. \end{cases}$$

Therefore the Courant-Weyl method of Dirichlet-Neumann-bracketing was used. This technique is not available for $p \neq 2$, but a partial analogue was established by Evans and Harris in [1] for Sobolev spaces $W_p^1(\Omega)$ on a wide class of domains. We want to transfer this idea and extend the results to the general case of Banach spaces $1 \leq p < \infty$.

References

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[3] H. TRIEBEL. Entropy and approximation numbers of limiting embeddings, an approach via Hardy inequalities and quadratic forms. *J. Approx. Theory*, 164(1): 31-46, 2012.