The Bohnenblust-Hille inequality: yesterday and today

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The multilinear Bohnenblust-Hille inequality shows that, given a real or complex matrix $(a_{i_1,\dots,i_m})_{(i_1,\dots,i_m)\in\{1,\dots,n\}^m}$, we have

$$\Big(\sum_{i_1,\dots,i_m} \big|a_{i_1,\dots,i_m}\big|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le C \sup_{\substack{\|(x_i^k)_{i=1}^n\|_\infty \le 1\\ 1\le k\le m}} \Big|\sum_{i_1,\dots,i_m} a_{i_1,\dots,i_m}\; x_{i_1}^1 \dots x_{i_m}^m\Big|,$$

where the constant $C = C(m)$ only depends on the degree m and not on n. Moreover, there is a polynomial counterpart: For each m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ in n variables $z_1, \dots, z_n$,

$$\Big(\sum_{|\alpha|=m} \big|c_\alpha\big|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le C \sup_{\|z\|_\infty \le 1} |P(z)|\,,$$

where the constant $C = C(m)$ again only depends on m.

Both inequalities were published by Bohnenblust and Hille in 1931 in the context of Dirichlet series, and for $m = 2$ they form Littlewood's so-called 4/3-inequalities which can be considered as forerunners of Grothendieck's inequality.

Recently, various authors, often with different motivation, improved and generalized these two scales of inequalities, and gave new interesting applications (even in quantum information theory). We plan to survey on some of these new developments.