THE LEIBNIZ FRACTIONAL RULE OF DIFFERENTIATION

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ABSTRACT. We present some recent results with S. Oh on the fractional Leibniz rule of differentiation combined with Hölder's inequality. Let Δ be the Laplacian on $\mathbf{R}^n$ and consider the classical Bessel potential $J^s = (1-\Delta)^{s/2}$ and Riesz potential $D^s = (-\Delta)^{s/2}$. We revisit the inequalities of Kato and Ponce concerning the L^r norm of the Bessel potential J^s (or the Riesz potential D^s) of the product of two functions in terms of the product of the L^p norm of one function and the L^q norm of the the Bessel potential J^s (resp. Riesz potential D^s) of the other function, i.e.,

$$\big\|J^s(fg)\big\|_{L^r} \le C\Big[\big\|f\big\|_{L^p}\big\|J^s g\big\|_{L^q} + \big\|J^s f\big\|_{L^p}\big\|g\big\|_{L^q}\Big]$$

and an analogous inequality with D^s in place of J^s. Here the indices p, q, and r are related as in Hölder's inequality $1/p+1/q = 1/r$ and they satisfy $1 \le p, q \le \infty$ and $1/2 \le r < \infty$ and $s > \frac{n}{r} - n$. Also the estimate is of weak-type when either p or q is equal to 1. In the case $r < 1$ we indicate via an example that when $s \le \frac{n}{r} - n$ the inequality fails. We explain how these problems can be addressed via analysis of multilinear multiplier operators. We also discuss extensions of these results to the multi-parameter case.

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